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SOLUTION OF INTEGRAL EQUATION VIA COMMON FIXED POINT RESULTS FOR (ψ, β)-GERAGHTY CONTRACTION TYPE MAPPING IN $b-$METRIC SPACES

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Abstract: The scope of this paper encompasses novel and extended results regarding a few common fixed points in complete $b-$metric spaces, with a particular focus on $(\psi, \beta)-$ Geraghty type contractive mapping. The study looks at real-world uses, discussing common fixed point results linked to integral type contractions and checking if solutions to integral equations exist.

Keywords and Phrases: Common fixed point, Geraghty contraction, $b-$metric spaces, $b-$Cauchy sequence, Integral equations.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

Over the last five decades, the study of fixed point(FP) theory has played a key role in addressing issues related to nonlinear phenomena. The evolution of FP theorems and the development of diverse techniques have significantly contributed to advancing both pure and applied analysis, as well as to the fields of topology and geometry.

In the year 1973, Geraghty [11] introduced a set of functions that extends the Banach contraction principle. This significant contribution aimed to provide a more versatile and comprehensive exploration, allowing researchers and mathematicians to extend their investigations beyond the traditional constraints of the Banach contraction principle. This exploration has opened new avenues for understanding the behaviour of dynamical systems and their stability. Consequently, researchers have been able to apply these extended concepts to various complex problems in mathematical physics and engineering, fostering a deeper comprehension of nonlinear dynamics. This extension has proven valuable in various mathematical contexts, fostering a deeper understanding of FP theorems and providing a more comprehensive perspective for mathematical analysis.

The idea of $b-$metric spaces(b-MS), initiated by Bakhtin [6] in 1989, served as a generalization of traditional metric spaces. Subsequently, numerous papers have been published on FP theory within these spaces. For a comprehensive exploration of the further work and results in b-MS, interested readers are directed to [4, 5, 7, 9, 10].

Definition 1.1. [9] *Consider a set $\mathcal{X}$ (which is non-empty) and a real number $\mathsf{s} \geq 1$. A function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is called a b-metric on $\mathcal{X}$ if, for all $\mathsf{x}, \mathsf{y}, \delta \in \mathcal{X}$, the following conditions hold:*

1. $\rho(\mathsf{x}, \mathsf{y}) = 0 \iff \mathsf{x} = \mathsf{y}$,
2. $\rho(\mathsf{x}, \mathsf{y}) = \rho(\mathsf{y}, \mathsf{x})$,
3. $\rho(\mathsf{x}, \mathsf{y}) \leq \mathsf{s}[\rho(\mathsf{x}, \delta) + \rho(\delta, \mathsf{y})]$.

Then, $(\mathcal{X}, \rho)$ is known as a b-MS with parameter s.

Example 1.2. Consider a metric space $(\mathcal{X}, \rho)$ with parameters $\beta > 1$, $\lambda \geq 0$. Define the function $\rho'(\mathsf{x}, \mathsf{y}) = \lambda\rho(\mathsf{x}, \mathsf{y}) + \rho(\mathsf{x}, \mathsf{y})^{\beta}$ for all $\mathsf{x}, \mathsf{y} \in \mathcal{X}$. The resulting space $(\mathcal{X}, \rho')$ is a b-MS with the parameter $\mathsf{s} = 2\beta - 1$ but does not qualify as a metric space on $\mathcal{X}$.

Definition 1.3. [1] *A sequence (x_n) in b-MS is called a b-Cauchy sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\rho(x_m, x_n) < \varepsilon$.*

Definition 1.4. [1] *A sequence (x_n) in b-MS is said to converge to $x \in \mathcal{X}$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$.*

Definition 1.5. [1] *A b-MS is said to be complete if every b-Cauchy sequence in $\mathcal{X}$ is convergent to a point in $\mathcal{X}$.*

Definition 1.6. [11] *Let Ω be the collection of all functions $\alpha : [0, \infty) \to [0, 1)$ that satisfy the condition:*

$$\lim_{\mathfrak{p}\to\infty} \alpha(t_{\mathfrak{p}}) = 1 \text{ implies } \lim_{\mathfrak{p}\to\infty} t_{\mathfrak{p}} = 0.$$

The Geraghty contraction, a theorem established by Geraghty, is expressed as follows.

Theorem 1.7.[17] *Consider a metric space $(\mathcal{X}, \rho)$ which is complete, and let $\mathtt{M} : \mathcal{X} \to \mathcal{X}$ be a mapping. Suppose there exists $\alpha \in \Omega$ such that for all $\mathsf{x}, \mathsf{y} \in \mathcal{X}$*

$$\rho(\mathtt{M}\mathsf{x}, \mathtt{M}\mathsf{y}) \leq \alpha(\rho(\mathsf{x}, \mathsf{y})) \cdot \rho(\mathsf{x}, \mathsf{y}).$$

Then $\mathtt{M}$ has a unique fixed point $z \in \mathcal{X}$.

In 2011, Dukic et al. [8] revisited Theorem 1.7, placing it within the context of $b-$metric spaces as detailed in [20].

Definition 1.8. [20] *Consider a b-MS $(\mathcal{X}, \rho)$ with a parameter $\mathsf{s} \geq 1$, and let Ω be the set of all functions $\alpha : [0, \infty) \to [0, \frac{1}{\mathsf{s}})$ that adhere to the following condition:*

$$\lim_{\mathfrak{p} \to \infty} \alpha(t_p) = \frac{1}{\mathsf{s}} \implies \lim_{\mathfrak{p} \to \infty} t_p = 0.$$

Theorem 1.9. [4] *Consider a complete b-MS $(\mathcal{X}, \rho)$ with a parameter $\mathsf{s} \geq 1$, and let $\mathtt{M} : \mathcal{X} \to \mathcal{X}$ be a mapping. Suppose that there exists $\beta \in \Omega$ satisfying:*

$$\rho(\mathtt{M}\mathsf{x}, \mathtt{M}\mathsf{y}) \leq \beta((\Theta(\mathsf{x}, \mathsf{y}))(\Theta(\mathsf{x}, \mathsf{y})), \ \forall \mathsf{x} \geq \mathsf{y},$$

where

$$\Theta(\mathsf{x}, \mathsf{y}) = \max\{\rho(\mathsf{x}, \mathsf{y}), \rho(\mathsf{x}, \mathtt{M}\mathsf{x}), \rho(\mathsf{y}, \mathtt{M}\mathsf{y}), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{x}, \mathtt{M}\mathsf{y}) + \rho(\mathsf{y}, \mathtt{M}\mathsf{x})\big)\},$$

for all $\mathsf{x}, \mathsf{y} \in \mathcal{X}$, then $\mathtt{M}$ has a unique fixed point $\mathsf{x}^* \in \mathcal{X}$.

Definition 1.10. [10] *Let $\mathcal{A}$ denote the family of all altering distance functions, i.e.,*

$$\mathcal{A} = \Big\{\psi : [0, \infty) \to [0, \infty) \ \Big| \ \psi \text{ is continuous, non-decreasing, and } \psi(t) = 0 \iff t = 0\Big\}.$$

A function $\psi \in \mathcal{A}$ is called an altering distance function if it satisfies:

1. *ψ is continuous and non-decreasing on $[0, \infty)$;*

2. *$\psi(t) = 0$ if and only if $t = 0$.*

Definition 1.11. [16] *Let $(\mathcal{X}, \preceq)$ be a partially ordered set($\mathcal{POSET}$), and let $\mathtt{M} : \mathcal{X} \to \mathcal{X}$ be a mapping. We say that $\mathtt{M}$ is monotone non-decreasing if for $\mathsf{x}, \mathsf{y} \in \mathcal{X}$, $\mathsf{x} \preceq \mathsf{y}$ implies $\mathtt{M}(\mathsf{x}) \preceq \mathtt{M}(\mathsf{y})$.*

Theorem 1.12. [16] *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$ and suppose that there is a metric ρ on $\mathcal{X}$ such that $(\mathcal{X}, \rho)$ is a metric space which is complete. Let $\mathtt{M} : \mathcal{X} \to \mathcal{X}$ be a monotone non-decreasing mapping such that there exists $s \in [0, 1)$ with*

$$\rho(\mathtt{M}\mathsf{x}, \mathtt{M}\mathsf{y}) \leq s\rho(\mathsf{x}, \mathsf{y}), \ \forall \mathsf{x} \preceq \mathsf{y},$$

and assume that either $\mathtt{M}$ is continuous or $\mathcal{X}$ is such that if there is a non-decreasing sequence $\{\mathsf{x}_\mathfrak{p}\} \to \mathsf{x}$, $\mathsf{x} \in \mathcal{X}$, then $\mathsf{x}_\mathfrak{p} \preceq \mathsf{x}$ or $\mathsf{x}_\mathfrak{p} \succeq \mathsf{x}$ for all $\mathfrak{p} \geq 1$. Moreover, if there is $\mathsf{x}_0 \in \mathcal{X}$ with $\mathsf{x}_0 \preceq \mathtt{M}\mathsf{x}_0$ or $\mathsf{x}_0 \succeq \mathtt{M}\mathsf{x}_0$, then $\mathtt{M}$ has a fixed point.

Theorem 1.13. [4] *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$, and suppose that there is a metric ρ on $\mathcal{X}$ such that $(\mathcal{X}, \rho)$ is a metric space which is complete. Let $\mathtt{M} : \mathcal{X} \to \mathcal{X}$ be an increasing mapping such that there is $\mathsf{x}_0 \in \mathcal{X}$ with $\mathsf{x}_0 \preceq \mathtt{M}(\mathsf{x}_0)$. Suppose that there exists $\alpha \in \Omega$ such that*

$$\rho(\mathtt{M}\mathsf{x}, \mathtt{M}\mathsf{y}) \leq \alpha(\rho(\mathsf{x}, \mathsf{y})) \cdot \rho(\mathsf{x}, \mathsf{y}) \quad \textit{for all } \mathsf{x}, \mathsf{y} \in \mathcal{X} \textit{ with } \mathsf{x} \preceq \mathsf{y},$$

and assume that either $\mathtt{M}$ is continuous or $\mathcal{X}$ is such that if there is an increasing sequence
$\{\mathsf{x}_\mathfrak{p}\} \to \mathsf{x}$, $\mathsf{x} \in \mathcal{X}$, then $\mathsf{x}_\mathfrak{p} \preceq \mathsf{x}$ for all $\mathfrak{p} \geq 1$. In addition, if for all $\mathsf{x}, \mathsf{y} \in \mathcal{X}$, there exists $z \in \mathcal{X}$ which is comparable to x and y, then $\mathtt{M}$ has a unique fixed point in $\mathcal{X}$.

Definition 1.14. [3] *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$ and $\mathtt{M}, \mathtt{N} : \mathcal{X} \to \mathcal{X}$. The pair $(\mathtt{M}, \mathtt{N})$ is said to be weakly increasing if $x \preceq y \Longrightarrow \mathtt{M}x \preceq \mathtt{N}y$ for all x, $y \in \mathcal{X}$.*

Lemma 1.15. [21] *If Ψ is an altering distance function and $\mathcal{Y} : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\Psi(t) > \mathcal{Y}(t)$ for all $t > 0$, then $\mathcal{Y}(0) = 0$.*

In recent years, there has been a notable trend among researchers to generalize Geraghty's result across different metric spaces. This paper contributes to this trend by extended some common fixed point($\mathcal{CFP}$) theorems specifically for (ψ, β)- Geraghty contractive mapping with in the framework of b-MS.

2. Main results

Theorem 2.1. *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$ and suppose that there exists a $b-$metric ρ on $\mathcal{X}$ such that $(\mathcal{X}, \rho)$ is a complete b-MS. Let $\mathtt{M}$ and $\mathtt{N}$ be weakly increasing mappings from $\mathcal{X}$ to itself. Suppose the following inequality holds for all $\mathsf{x} \succeq \mathsf{y}$,*

$$\psi(\rho(\mathtt{M}\mathsf{x}, \mathtt{N}\mathsf{y})) \leq \beta(\Theta(\mathsf{x}, \mathsf{y}))\alpha(\Theta(\mathsf{x}, \mathsf{y})), \ \textit{for all } \mathsf{x} \succeq \mathsf{y}, \tag{2.1}$$

where

$$\Theta(\mathsf{x}, \mathsf{y}) = \max\{\rho(\mathsf{x}, \mathsf{y}), \rho(\mathsf{x}, \mathtt{M}\mathsf{x}), \rho(\mathsf{y}, \mathtt{N}\mathsf{y}), \frac{1}{2\mathtt{s}}\big(\rho(\mathsf{x}, \mathtt{N}\mathsf{y}) + \rho(\mathsf{y}, \mathtt{M}\mathsf{x})\big)\},$$

and $\beta \in \Omega$, $\psi \in \Psi$, and $\alpha : [0, \infty) \to [0, \frac{1}{\mathsf{s}})$ is a continuous function with the condition $\psi(t) > \alpha(t)$, for all $t > 0$. Furthermore, assume that for each pair of elements $\mathsf{x}, \mathsf{y} \in \mathcal{X}$, there is $z \in \mathcal{X}$ which is comparable to both x and y. Assume also that at least one of the mappings M or N is continuous, that is, for any sequence $\{x_n\} \subset X$ with $x_n \to x$ (with respect to ρ), we have $Mx_n \to Mx$ (respectively $Nx_n \to Nx$). Then $\mathtt{M}$ and $\mathtt{N}$ possess a unique common fixed point.

Proof. Assume $\mathsf{x}_0 \in \mathcal{X}$ is an arbitrary point in $\mathcal{X}$ such that $\mathtt{M}\mathsf{x}_0 = \mathsf{x}_1$ and $\mathtt{N}\mathsf{x}_1 = \mathsf{x}_2$. Continuing in this manner, the sequences $\{\mathsf{x}_\mathfrak{p}\}$ and $\{\mathsf{y}_\mathfrak{p}\}$ in $\mathcal{X}$ can be constructed as follows:

$$\mathsf{x}_{2\mathfrak{p}+1} = \mathtt{M}\mathsf{x}_{2\mathfrak{p}} = \mathsf{y}_{2\mathfrak{p}}, \quad \mathsf{x}_{2\mathfrak{p}+2} = \mathtt{N}\mathsf{x}_{2\mathfrak{p}+1} = \mathsf{y}_{2\mathfrak{p}+1}, \quad \forall \mathfrak{p} \in \mathbb{N}. \tag{2.2}$$

As $\mathtt{M}$ and $\mathtt{N}$ are weakly increasing functions, we have

$$\mathsf{x}_1 \preceq \mathsf{x}_2 \preceq \mathsf{x}_3 \cdots \preceq \mathsf{x}_{2\mathfrak{p}+1} \preceq \mathsf{x}_{2\mathfrak{p}+2} \ldots$$

Thus,

$$\mathsf{y}_0 \preceq \mathsf{y}_1 \preceq \mathsf{y}_2 \cdots \preceq \mathsf{y}_{2\mathfrak{p}} \preceq \mathsf{y}_{2\mathfrak{p}+1} \ldots$$

Assume that there is $\mathfrak{p} \in \mathbb{N}$ such that $\mathsf{y}_{2\mathfrak{p}-1} = \mathsf{y}_{2\mathfrak{p}}$. We show that $\{\mathsf{x}_\mathfrak{p}\}$ is a Cauchy sequence.

$$\psi(\rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1})) = \psi(\rho(\mathtt{M}\mathsf{x}_{2\mathfrak{p}}, \mathtt{N}\mathsf{x}_{2\mathfrak{p}+1})) \leq \beta(\Theta(\mathsf{x}_{2\mathfrak{p}}, \mathsf{x}_{2\mathfrak{p}+1}))\alpha(\Theta(\mathsf{x}_{2\mathfrak{p}}, \mathsf{x}_{2\mathfrak{p}+1}))$$

where

$$\begin{aligned}
&\Theta(\mathsf{x}_{2\mathfrak{p}}, \mathsf{x}_{2\mathfrak{p}+1}) \\
&\quad = \max\{\rho(\mathsf{x}_{2\mathfrak{p}}, \mathsf{x}_{2\mathfrak{p}+1}), \rho(\mathsf{x}_{2\mathfrak{p}}, \mathtt{M}\mathsf{x}_{2\mathfrak{p}}), \rho(\mathsf{x}_{2\mathfrak{p}+1}, \mathtt{N}\mathsf{x}_{2\mathfrak{p}+1}), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{x}_{2\mathfrak{p}}, \mathtt{N}\mathsf{x}_{2\mathfrak{p}+1}) + \rho(\mathsf{x}_{2\mathfrak{p}+1}, \mathtt{M}\mathsf{x}_{2\mathfrak{p}})\big)\} \\
&\quad = \max\{\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1}), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}+1}) + \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}})\big)\} \\
&\quad \leq \max\{\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1}), \frac{\mathsf{s}}{2\mathsf{s}}\big(\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}) + \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1})\big)\} \\
&\quad = \max\{\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1}), \frac{1}{2}\big(\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}) + \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1})\big)\} \\
&\quad = \max\{\rho(\mathsf{y}_{2\mathfrak{p}-1}, \mathsf{y}_{2\mathfrak{p}}), \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1})\} = 0.
\end{aligned}$$

This implies

$$\psi(\rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}+1})) = 0.$$

From this, it follows that $\mathsf{y}_{2\mathfrak{p}+1} = \mathsf{y}_{2\mathfrak{p}}$. Therefore, $\mathsf{y}_m = \mathsf{y}_{2\mathfrak{p}-1}$ holds for all $m \geq 2\mathfrak{p}$. As a result, for all $m \geq 2\mathfrak{p}$, we get $\mathsf{x}_m = \mathsf{x}_{2\mathfrak{p}}$. Hence the sequence $\{\mathsf{x}_\mathfrak{p}\}$ is a Cauchy

sequence.
As a second consideration, let us assume $y_{\mathfrak{p}} \neq y_{\mathfrak{p}+1}$ for all $\mathfrak{p} \geq 1$.
Define $\Delta_{\mathfrak{p}} = \rho(y_{\mathfrak{p}}, y_{\mathfrak{p}+1})$. Now, we aim to prove that $\Delta_{\mathfrak{p}} \to 0$ as $\mathfrak{p} \to \infty$. As $x_{2\mathfrak{p}}$ and $x_{2\mathfrak{p}+1}$ are comparable, we can deduce again from Equation (2.1),

$$\psi(\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1})) = \psi(\rho(Mx_{2\mathfrak{p}+2}, Nx_{2\mathfrak{p}+1})) \leq \beta(\Theta(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1}))\alpha(\Theta(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1})) \tag{2.3}$$

where

$$\begin{aligned}\Theta(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1}) &= \max\{\rho(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1}), \rho(x_{2\mathfrak{p}+2}, Mx_{2\mathfrak{p}+2}), \rho(x_{2\mathfrak{p}+1}, Nx_{2\mathfrak{p}+1}),\\ &\quad \frac{1}{2s}\left(\rho(x_{2\mathfrak{p}+2}, Nx_{2\mathfrak{p}+1}) + \rho(x_{2\mathfrak{p}+1}, Mx_{2\mathfrak{p}+2})\right)\}\\ &= \max\{\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}), \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2}), \rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}), \frac{1}{2s}\left(\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+1}) + \rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+2})\right)\}\\ &\leq \max\{\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}), \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2}), \rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}), \frac{s}{2s}\left(\rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}) + \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})\right)\}\\ &= \max\{\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}), \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2}), \rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}), \frac{1}{2}\left(\rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}) + \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})\right)\}\\ &= \max\{\rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}), \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})\}.\end{aligned}$$

If $\rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}+1}) \leq \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})$, then $\Theta(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1}) = \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})$. According to Equation (2.3), we obtain

$$\psi(\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1})) \leq \beta(\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2}))\alpha(\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}+2})).$$

By Equation (2.1), we obtain

$$\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1}) \leq \frac{1}{s}\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1}), \mathfrak{p} \in \mathbb{N}.$$

This is a contradiction. Hence, we conclude

$$\Theta(x_{2\mathfrak{p}+2}, x_{2\mathfrak{p}+1}) = \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}). \tag{2.4}$$

Subsequently, using Equation (2.3), we derive

$$\psi(\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1})) \leq \beta(\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}))\alpha(\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}})). \tag{2.5}$$

By Equation (2.1), we obtain

$$\rho(y_{2\mathfrak{p}+2}, y_{2\mathfrak{p}+1}) \leq \rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}), \mathfrak{p} \in \mathbb{N}. \tag{2.6}$$

Similarly,

$$\rho(y_{2\mathfrak{p}+1}, y_{2\mathfrak{p}}) \leq \rho(y_{2\mathfrak{p}}, y_{2\mathfrak{p}-1}), \mathfrak{p} \in \mathbb{N}. \tag{2.7}$$

By combining Equation (2.6) and Equation (2.7), we obtain

$$\rho(\mathsf{y}_{2\mathfrak{p}+2}, \mathsf{y}_{2\mathfrak{p}+1}) \leq \rho(\mathsf{y}_{2\mathfrak{p}+1}, \mathsf{y}_{2\mathfrak{p}}) \leq \rho(\mathsf{y}_{2\mathfrak{p}}, \mathsf{y}_{2\mathfrak{p}-1}), \mathfrak{p} \in \mathbb{N}.$$

Consequently, the sequence $\{\Delta_{\mathfrak{p}}\}$ decreases monotonically, therefore there is $r \geq 0$ such that

$$\lim_{\mathfrak{p}\to\infty} \Delta_{\mathfrak{p}} = r. \tag{2.8}$$

As $\mathfrak{p} \to \infty$ in Equation (2.5) and applying Equation (2.8), we obtain $\psi(r) \leq \alpha(r)$, for $\beta \in \Omega$. This contradicts the statement of Theorem 2.1. Thus, $r = 0$.
This implies that $\Delta_{\mathfrak{p}} \to 0$ as $\mathfrak{p} \to \infty$.
Next, we show that $\{\mathsf{x}_{\mathfrak{p}}\}$ is a Cauchy sequence. To demonstrate this, our objective is to establish the Cauchy property for $\{\mathsf{x}_{2\mathfrak{p}}\}$. Assuming the contrary, let us suppose that $\{\mathsf{x}_{2\mathfrak{p}}\}$ is not a Cauchy sequence. Consequently, for any $\varepsilon > 0$, there are $\mathfrak{p}_{\mathsf{k}}$, $\mathfrak{q}_{\mathsf{k}} \in \mathbb{N} \cup \{0\}$ with the property $\mathfrak{p}_{\mathsf{k}} > \mathfrak{q}_{\mathsf{k}} > \mathsf{k}$ for all $\mathsf{k} > 0$,

$$\rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}) > \varepsilon \text{ and } \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}-1}}) < \varepsilon. \tag{2.9}$$

Utilizing Equation (2.9) and applying the b-triangle inequality, we have

$$\begin{aligned} &\varepsilon < \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}) \leq \mathsf{s}(\rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}-1}}) + \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}-1}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}})), \\ &\frac{\varepsilon}{\mathsf{s}} \leq \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}-1}}) + \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}-1}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}). \end{aligned}$$

As $\mathsf{k} \to \infty$ in the above condition, we obtain

$$\lim_{\mathsf{k}\to\infty} \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}) = \frac{\varepsilon}{\mathsf{s}}.$$

Again applying the b-triangle inequality, we have

$$\rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}-1}}) \leq \mathsf{s}(\rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}) + \rho(\mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}-1}})).$$

As $\mathsf{k} \to \infty$ in the above condition, we obtain

$$\lim_{\mathsf{k}\to\infty} \rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}-1}}) = \frac{\varepsilon}{\mathsf{s}}.$$

Now,

$$\rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}}) \leq \mathsf{s}(\rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}}}, \mathsf{x}_{2\mathfrak{q}_{\mathsf{k}+1}}) + \rho(\mathsf{x}_{2\mathfrak{q}_{\mathsf{k}+1}}, \mathsf{x}_{2\mathfrak{p}_{\mathsf{k}}})).$$

Noting that $x_{2q_k+1} = Mx_{2q_k}$ and $x_{2p_k} = Nx_{2p_k+1}$, we can write

$$\rho(x_{2q_k}, x_{2p_k}) \leq s\left(\rho(x_{2q_k}, x_{2q_k+1}) + \rho(Mx_{2q_k}, Nx_{2p_k+1})\right).$$

As $\mathsf{k} \to \infty$, we have

$$\frac{\varepsilon}{\mathsf{s}} \leq \lim_{k\to\infty}(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_\mathsf{k}+1})).$$

As ψ is both continuous and non-decreasing, it follows that

$$\psi(\frac{\varepsilon}{\mathsf{s}}) \leq \lim_{k\to\infty}\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_\mathsf{k}+1})). \tag{2.10}$$

From Equation (2.1), we have

$$\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}})) \leq \beta(\Theta(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}))\delta(\Theta(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}})), \tag{2.11}$$

where

$$\begin{aligned}\Theta(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}) &= \max\{\rho(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}), \rho(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}), \rho(\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}),\\
&\quad \frac{1}{2\mathsf{s}}(\rho(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}) + \rho(\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}, \mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}))\}\\
&= \max\{\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}), \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_y), \rho(\mathsf{y}_{2\mathsf{p}_\mathsf{k}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}), \frac{1}{2\mathsf{s}}(\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}) + \rho(\mathsf{y}_{2\mathsf{p}_k}, \mathsf{y}_{2\mathsf{q}_\mathsf{k}}))\}\\
&\leq \max\{\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}), \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{q}_\mathsf{k}}), \rho(\mathsf{y}_{2\mathsf{p}_\mathsf{k}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}),\\
&\qquad \frac{\mathsf{s}}{2\mathsf{s}}(\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}) + \rho(\mathsf{y}_{2\mathsf{p}_\mathsf{k}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}) + \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}) + \rho(\mathsf{y}_{2\mathsf{q}_\mathsf{k}}, \mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}))\}\\
&= \max\{\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}), \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{q}_\mathsf{k}}), \rho(\mathsf{y}_{2\mathsf{p}_\mathsf{k}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}),\\
&\qquad \frac{1}{2}(\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}) + \rho(\mathsf{y}_{2\mathsf{p}_\mathsf{k}}, \mathsf{y}_{2\mathsf{p}_{\mathsf{k}+1}}) + \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}) + \rho(\mathsf{y}_{2\mathsf{q}_\mathsf{k}}, \mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}))\}.\end{aligned}$$

From Equation (2.4), we have

$$\Theta(\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}}) \leq \rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}).$$

Following Equation (2.11), it can be deduced that

$$\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}})) \leq \beta(\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}}))\alpha(\rho(\mathsf{y}_{2\mathsf{q}_{\mathsf{k}-1}}, \mathsf{y}_{2\mathsf{p}_\mathsf{k}})). \tag{2.12}$$

Repeating the limit process as $\mathsf{k} \to \infty$ in Equation (2.12) and considering the property $\beta \in \Omega$, we obtain

$$\lim_{\mathsf{k}\to\infty}\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}})) < \alpha(\frac{\varepsilon}{\mathsf{s}}).$$

Hence from Equation (2.10), we get

$$\psi(\frac{\varepsilon}{\mathsf{s}}) \leq \lim_{\mathsf{k}\to\infty}\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathsf{q}_\mathsf{k}}, \mathtt{N}\mathsf{x}_{2\mathsf{p}_{\mathsf{k}+1}})) \leq \alpha(\frac{\varepsilon}{\mathsf{s}}).$$

This is possible only if $\varepsilon = 0$. This is a contradiction. Therefore, $\{\mathtt{x}_{2\mathfrak{p}}\}$ is a $b-$Cauchy sequence, implying that $\{\mathtt{x}_{\mathfrak{p}}\}$ is also a $b-$Cauchy sequence for all $\mathfrak{p} \geq 1$. Hence, there is $\omega \in \mathcal{X}$ such that

$$\lim_{\mathfrak{p}\to\infty} \mathtt{x}_{\mathfrak{p}} = \omega.$$

There after, we prove that ω is a fixed point of $\mathtt{M}$.
Due to the continuity of $\mathtt{M}$ and the convergence $\mathtt{x}_{2\mathfrak{p}+1} \to \omega$, it can be concluded that

$$\omega = \lim_{\mathfrak{p}\to\infty} \mathtt{x}_{2\mathfrak{p}+1} = \lim_{\mathfrak{p}\to\infty} \mathtt{M}\mathtt{x}_{2\mathfrak{p}} = \mathtt{M}\omega.$$

Thus, ω is a fixed point of $\mathtt{M}$.
Also,

$$\psi(\rho(\omega, \mathtt{N}\omega)) = \psi((\mathtt{M}\omega, \mathtt{N}\omega)) \leq \beta(\Theta(\omega, \omega))\alpha(\Theta(\omega, \omega)) \tag{2.13}$$

where

$$\begin{aligned}\Theta(\omega, \omega) &= \max\{\rho(\omega, \omega), \rho(\omega, \mathtt{M}\omega), \rho(\omega, \mathtt{N}\omega), \frac{1}{2\mathtt{s}}\big(\rho(\omega, \mathtt{N}\omega) + \rho(\omega, \mathtt{M}\omega)\big)\}\\ &\leq \rho(\omega, \mathtt{N}\omega).\end{aligned}$$

Then, from Equation (2.13), we get

$$\psi(\rho(\omega, \mathtt{N}\omega)) = \psi((\mathtt{M}\omega, \mathtt{N}\omega)) \leq \beta(\rho(\omega, \mathtt{N}\omega))\alpha(\rho(\omega, \mathtt{N}\omega)).$$

Consequently, $\psi(\frac{1}{\mathtt{s}}) \leq \lim_{k\to\infty} \psi(\rho(\omega, \mathtt{N}\omega)) \leq \alpha(\frac{1}{\mathtt{s}})$.

Hence $\mathtt{N}\omega = \omega$. Therefore, $\mathtt{M}\omega = \mathtt{N}\omega = \omega$. That is, ω is a $\mathcal{CFP}$ of $\mathtt{M}$ and $\mathtt{N}$.

Next, we demand that $\mathcal{CFP}$ of $\mathtt{M}$ and $\mathtt{N}$ is unique. Assume on the contrary that $\mathtt{M}\omega = \mathtt{N}\omega = \omega$ and $\mathtt{M}\varpi = \mathtt{N}\varpi = \varpi$ but $\omega \neq \varpi$. As per the assumption, we can substitute $\mathtt{x}$ with ω and $\mathtt{y}$ with ϖ into Equation (2.1), yielding

$$\psi(\rho(\omega, \varpi)) = \psi(\rho(\mathtt{M}\omega, \mathtt{N}\varpi)) \leq \alpha(\rho(\omega, \varpi))\beta(\rho(\omega, \varpi)) < \beta(\rho(\omega, \varpi)).$$

Applying the statement of Theorem 2.1 and Lemma 1.15, we get $\rho(\omega, \varpi) = 0$. It is possible only if $\omega = \varpi$. Thus, $\mathtt{M}$ and $\mathtt{N}$ have a unique common fixed point.

Definition 2.2. [15] *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$. We say that $(X, \preceq)$ is regular if for every non-decreasing sequence $\{x_p\}_{p\in\mathbb{N}}$ in X (i.e. $x_p \preceq x_{p+1}$ for all p) that converges to some $z \in X$, we have $x_p \preceq z$ for every $p \in \mathbb{N}$.*

Theorem 2.3. *Let $(\mathcal{X}, \preceq)$ be a $\mathcal{POSET}$ and suppose that there exists a $b-$metric ρ*

on $\mathcal{X}$ such that $(\mathcal{X}, \rho)$ is regular. Let $\mathtt{M}, \mathtt{N} : \mathcal{X} \to \mathcal{X}$ be weakly increasing mappings, satisfying

$$\psi(\rho(\mathtt{M}\mathsf{x}, \mathtt{N}\mathsf{y})) \leq \alpha(\Theta(\mathsf{x}, \mathsf{y}))\beta(\Theta(\mathsf{x}, \mathsf{y})), \quad \forall \mathsf{x} \succeq \mathsf{y},$$

where

$$\Theta(\mathsf{x}, \mathsf{y}) = \max\{\rho(\mathsf{x}, \mathsf{y}), \rho(\mathsf{x}, \mathtt{M}\mathsf{x}), \rho(\mathsf{y}, \mathtt{N}\mathsf{y}), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{x}, \mathtt{N}\mathsf{y}) + \rho(\mathsf{y}, \mathtt{M}\mathsf{x})\big)\},$$

and α belongs to Ω*, ψ* belongs to Ψ*, and $\beta : [0, \infty) \to [0, \frac{1}{\mathsf{s}})$ is a continuous function with the condition $\psi(t) > \beta(t)$ for all $t > 0$.*

Furthermore, assume that for each pair of elements $\mathsf{x}, \mathsf{y} \in \mathcal{X}$, there is $z \in \mathcal{X}$ that is comparable to both x and y. Assume also that at least one of the mappings M or N is continuous, that is, for any sequence $\{x_n\} \subset X$ with $x_n \to x$ (with respect to ρ), we have $Mx_n \to Mx$ (respectively $Nx_n \to Nx$). Then $\mathtt{M}$ and $\mathtt{N}$ possess a unique common fixed point.

Proof. Here, we introduce the same sequences $\{\mathsf{x}_\mathfrak{p}\}$ and $\{\mathsf{y}_\mathfrak{p}\}$ as employed in the proof of Theorem 2.1. It is evident that $\{\mathsf{x}_\mathfrak{p}\}$ constitutes a b-Cauchy sequence in $\mathcal{X}$, implying the existence of $\omega \in \mathcal{X}$ such that

$$\lim_{\mathfrak{p} \to \infty} \mathsf{x}_\mathfrak{p} = \omega. \tag{2.14}$$

As $\mathcal{X}$ is regular, there is a non-decreasing sequence $\{\mathsf{x}_\mathfrak{p}\}$ in $\mathcal{X}$ such that

$$\mathsf{x}_\mathfrak{p} \preceq \omega, \quad \forall \mathfrak{p} \in \mathbb{N}.$$

Therefore, $\mathsf{x}_\mathfrak{p}$ and ω are comparable. Furthermore, by considering the limit as $\mathfrak{p} \to \infty$ in Equation (2.2) and utilizing Equation (2.14), we obtain

$$\lim_{\mathfrak{p} \to \infty} \mathtt{M}(\mathsf{x}_{2\mathfrak{p}}) = \lim_{\mathfrak{p} \to \infty} \mathsf{x}_{2\mathfrak{p}+1} = \omega; \quad \lim_{\mathfrak{p} \to \infty} \mathtt{N}(\mathsf{x}_{2\mathfrak{p}+1}) = \lim_{\mathfrak{p} \to \infty} \mathsf{x}_{2\mathfrak{p}+2} = \omega. \tag{2.15}$$

Taking $\mathsf{x} = \mathsf{x}_{2\mathfrak{p}}$ and $\mathsf{y} = \omega$ in Equation (2.1), we have

$$\psi(\rho(\mathtt{M}\mathsf{x}_{2\mathfrak{p}}, \mathtt{N}\omega)) \leq \alpha(\Theta(\mathsf{x}_{2\mathfrak{p}}, \omega))\beta(\Theta(\mathsf{x}_{2\mathfrak{p}}, \omega)), \tag{2.16}$$

where

$$\begin{aligned}\Theta(\mathsf{x}_{2\mathfrak{p}}, \omega) &= \max\{\rho(\mathsf{x}_{2\mathfrak{p}}, \omega), \rho(\mathsf{x}_{2\mathfrak{p}}, \mathtt{M}\mathsf{x}_{2\mathfrak{p}}), \rho(\omega, \mathtt{N}\omega), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{x}_{2\mathfrak{p}}, \mathtt{N}\omega) + \rho(\omega, \mathtt{M}\mathsf{x}_{2\mathfrak{p}})\big)\} \\ &= \max\{\rho(\mathsf{x}_{2\mathfrak{p}}, \omega), \rho(\mathsf{x}_{2\mathfrak{p}}, \mathsf{x}_{2\mathfrak{p}+1}), \rho(\omega, \mathtt{N}\omega), \frac{1}{2\mathsf{s}}\big(\rho(\mathsf{x}_{2\mathfrak{p}}, \mathtt{N}\omega) + \rho(\omega, \mathsf{x}_{2\mathfrak{p}+1})\big)\}.\end{aligned}$$

As $\mathfrak{p} \to \infty$ in the above inequality and applying Equation (2.14) and Equation (2.15), we obtain

$$\begin{aligned}\Theta(\omega,\omega) &= \max\{\rho(\omega,\omega),\rho(\omega,\omega),\rho(\omega,\mathtt{N}\omega),\frac{1}{2\mathsf{s}}\big(\rho(\omega,\mathtt{N}\omega)+\rho(\omega,\omega)\big)\}\\ &= \max\{0,0,\rho(\omega,\mathtt{N}\omega),\frac{1}{2\mathsf{s}}\big(\rho(\omega,\mathtt{N}\omega)+0\big)\} = \rho(\omega,\mathtt{N}\omega).\end{aligned}$$

Subsequently, by Equation (2.16) and taking the limit as $\mathfrak{p} \to \infty$, we have

$$\psi(\rho(\omega,\mathtt{N}\omega)) \leq \alpha(\rho(\omega,\mathtt{N}\omega))\beta(\rho(\omega,\mathtt{N}\omega)).$$

Using Equations (2.14) and (2.15), we derive

$$\psi(\rho(\omega,\mathtt{N}\omega)) = 0.$$

Consequently, we have $\omega = \mathtt{N}\omega$.

If we take $\mathsf{x} = \omega$ and $\mathsf{y} = \mathsf{x}_{2\mathfrak{p}+1}$ in Equation (2.1), then the outcome is

$$\psi(\rho(\mathtt{M}\omega,\mathtt{N}\mathsf{x}_{2\mathfrak{p}+1})) \leq \alpha(\Theta(\omega,\mathsf{x}_{2\mathfrak{p}+1}))\beta(\Theta(\omega,\mathsf{x}_{2\mathfrak{p}+1})), \tag{2.17}$$

where

$$\begin{aligned}\Theta(\omega,\mathsf{x}_{2\mathfrak{p}+1}) &= \max\{\rho(\omega,\mathsf{x}_{2\mathfrak{p}+1}),\rho(\omega,\mathtt{M}\omega),\rho(\mathsf{x}_{2\mathfrak{p}+1},\mathtt{N}\mathsf{x}_{2\mathfrak{p}+1}),\frac{1}{2\mathsf{s}}\big(\rho(\omega,\mathtt{N}\mathsf{x}_{2\mathfrak{p}+1})+\rho(\mathsf{x}_{2\mathfrak{p}+1},\mathtt{M}\omega)\big)\}\\ &= \max\{\rho(\omega,\mathsf{x}_{2\mathfrak{p}+1}),\rho(\omega,\mathtt{M}\omega),\rho(\mathsf{x}_{2\mathfrak{p}+1},\mathsf{x}_{2\mathfrak{p}+2}),\frac{1}{2\mathsf{s}}\big(\rho(\omega,\mathsf{x}_{2\mathfrak{p}+2})+\rho(\mathsf{x}_{2\mathfrak{p}+1},\mathtt{M}\omega)\big)\}.\end{aligned}$$

As $\mathfrak{p} \to \infty$ in the above inequality and applying (2.14) and (2.15), we obtain

$$\begin{aligned}\Theta(\omega,\omega) &= \max\{\rho(\omega,\omega),\rho(\omega,\mathtt{M}\omega),\rho(\omega,\omega),\frac{1}{2\mathsf{s}}\big(\rho(\omega,\omega)+\rho(\omega,\mathtt{M}\omega)\big)\}\\ &= \max\{0,\rho(\omega,\mathtt{M}\omega),0,\frac{1}{2\mathsf{s}}\big(0+\rho(\omega,\mathtt{M}\omega)\big)\} = \rho(\omega,\mathtt{M}\omega).\end{aligned}$$

Subsequently, by Equation (2.17) and taking the limit as $\mathfrak{p} \to \infty$, we have

$$\psi(\rho(\mathtt{M}\omega,\omega)) \leq \alpha(\rho(\omega,\mathtt{M}\omega))\beta(\rho(\omega,\mathtt{M}\omega)).$$

Using Equations (2.14) and (2.15), we derive

$$\psi(\rho(\mathtt{M}\omega,\omega)) = 0.$$

Thus, we have $\omega = \mathtt{M}\omega = \mathtt{N}\omega$. Therefore, ω is a $\mathcal{CFP}$ of $\mathtt{M}$ and $\mathtt{N}$. Uniqueness of the fixed point follows from Theorem 2.1. This completes the proof.

3. Applications

As part of our applications, we explore Subsection 3.1, presenting several fixed point theorems designed for contractions of integral type.

3.1. Conclusions for fixed point solutions for mapping satisfying a contraction of integral type

The focus of this section is to demonstrate fixed point results for mapping adhering to a contraction of integral type in a complete ordered b-MS, we first introduce some notations before presenting the proofs.

Consider the set χ comprising functions $\phi : [0, \infty) \to [0, \infty)$ that adhere to:

1. For every compact subset of $[0, \infty)$, the function ϕ is Lebesgue integrable,
2. for every $\epsilon > 0$,

$$\int_0^{\infty} \phi(t)\, dt < \epsilon.$$

Consider a fixed positive integer $N \in \mathbb{N}^*$. Let $\phi_i, 1 \leq i \leq N$ be a collection belonging to χ. For all $t \geq 0$, we have

$$\begin{aligned}
I_1(t) &= \int_0^t \phi_1(\mathtt{s})d\mathtt{s},\\
I_2(t) &= \int_0^{I_1(t)} \phi_2(\mathtt{s})d\mathtt{s} = \int_0^{\int_0^{\mathtt{s}} \phi_1(\mathtt{s})d\mathtt{s}} \phi_2(\mathtt{s})d\mathtt{s},\\
I_3(t) &= \int_0^{I_2(t)} \phi_3(\mathtt{s})d\mathtt{s} = \int_0^{\int_0^{\mathtt{s}} \phi_2(\mathtt{s})d\mathtt{s}} \phi_3(\mathtt{s})d\mathtt{s},\\
&\vdots\\
\mathcal{I}_{\mathcal{N}-1}(t) &= \int_0^{\mathcal{I}_{\mathcal{N}-2}(t)} \phi_{\mathcal{N}-1}(\mathtt{s})d\mathtt{s} = \int_0^{\int_0^{\mathtt{s}} \phi_{\mathcal{N}-2}(\mathtt{s})d\mathtt{s}} \phi_{\mathcal{N}-1}(\mathtt{s})d\mathtt{s},\\
I_{\mathcal{N}}(t) &= \int_0^{\mathcal{I}_{\mathcal{N}-1}(t)} \phi_{\mathcal{N}}(\mathtt{s})d\mathtt{s}.
\end{aligned}$$

Now we derive the following theorem.

Theorem 3.1. *Suppose $\mathcal{X}$ is a $\mathcal{POSET}$ with a metric ρ making $(\mathcal{X}, \rho)$ a complete b-MS. Let $\mathtt{M}, \mathtt{N} : \mathcal{X} \to \mathcal{X}$ be continuous and weakly increasing mappings, satisfying the inequality:*

$$\mathcal{I}_N(\psi(\Theta(\mathtt{Mx}, \mathtt{Ny}))) \leq \alpha(\Theta(\mathsf{x}, \mathsf{y}))\mathcal{I}_N(\beta(\Theta(\mathsf{x}, \mathsf{y}))),\ \forall \mathsf{x} \succeq \mathsf{y}, \tag{3.1}$$

where

$$\Theta(\mathtt{x},\mathtt{y}) = \max\{\rho(\mathtt{x},\mathtt{y}), \rho(\mathtt{x},\mathtt{Mx}), \rho(\mathtt{y},\mathtt{Ny}), \frac{1}{2\mathtt{s}}\big(\rho(\mathtt{x},\mathtt{Ny}) + \rho(\mathtt{y},\mathtt{Mx})\big)\},$$

and α belongs to Ω, ψ belongs to Ψ, *and* $\beta : [0,\infty) \to [0, \frac{1}{\mathtt{s}})$ *is a continuous function with the condition* $\psi(t) > \beta(t)$, $\forall\ t > 0$.

Furthermore, assume that for each pair of elements $\mathtt{x},\mathtt{y} \in \mathcal{X}$*, there is* $z \in \mathcal{X}$ *that is comparable to both* $\mathtt{x}$ *and* $\mathtt{y}$*. If either* $\mathtt{M}$ *or* $\mathtt{N}$ *is continuous, then* $\mathtt{M}$ *and* $\mathtt{N}$ *possess a unique common fixed point.*

Proof. Let us define $\psi_1 = I_N \circ \psi$ and $\beta_1 = I_N \circ \beta$. Consequently, according to Equation (3.1), we can express:

$$\psi_1(\Theta(\mathtt{Mx},\mathtt{Ny})) \leq \alpha(\Theta(\mathtt{x},\mathtt{y}))\beta_1(\Theta(\mathtt{x},\mathtt{y})), \quad \forall \mathtt{x} \succeq \mathtt{y}.$$

As we know that the composition of continuous functions remains continuous ensures that ψ_1 and β_1 are both continuous. By invoking Theorem 2.1, we derive the desired outcome.

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